STRESS – STRAIN STATE AT THE APEX OF A SLIT IN THE ANTIPLANE STRAIN OF AN ELASTIC – PLASTIC SOLID

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Solutions describing arbitrarily large values of the stresses and strains at the apex of a slit (solution with a singularity) are considered in the analysis of the stress–strain state of a mountain mass in the neighborhood of the slit by using an elastic model. An elastic–plastic model can be considered in place of the elastic model which results in such solutions. When the elastic–plastic model is chosen correctly, a more exact description of the real values of the stresses and strains at the slit apex can be expected. The strains must be evaluated in this problem for which an exact solution in stresses is known for the elastic–plastic model [1]. The exact solution presented describes the stress–strain state without singularities and yields an interpretation of the ultimate stresses (forces) as adhesion forces hindering the expansion of the slit. This permits the proposal of a strength criterion for a material with shear strains. The solution can be suitable for estimating the size of the focus and energy when studying mountain shocks or shallow-focus earthquakes.

1. Let an elastic medium be slit along the half-plane xz ($x < 0$) so that only the displacement component W (the projection on the z axis) will be different from zero on the edges of the slit, and will take on values of equal magnitude but opposite sign on the slit edges. The displacement equals zero for $x > 0$ on the $y=0$ axis, the y axis is perpendicular to the plane of the slit, and the origin passes through the apex of the slit [1].

The strains ε_{xz}, ε_{yz} are related to the displacement

$$\varepsilon_{xz} = \frac{\partial W}{\partial x}, \ \varepsilon_{yz} = \frac{\partial W}{\partial y}, \ \frac{\partial \varepsilon_{yz}}{\partial x} = \frac{\partial \varepsilon_{xy}}{\partial y} \tag{1.1}$$

and determine the pure shear state; the principal strains are

$$\varepsilon_1 = \Gamma/2, \ \varepsilon_2 = 0, \ \varepsilon_3 = -\Gamma/2, \ \Gamma = \sqrt{\varepsilon_{xz}^2 + \varepsilon_{yz}^2} \tag{1.2}$$

where Γ is the principal shear on areas passing through the second principal direction

$$\varepsilon_{xz} / \varepsilon_{yz} = -\operatorname{tg} \psi(x, y) \tag{1.3}$$

and ψ is the angle made by the second principal direction with the x axis; the second principal direction is perpendicular to the z axis.

The principal stresses are

$$\sigma_1 = T, \ \sigma_2 = 0, \ \sigma_3 = -T, \ T = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} \tag{1.4}$$

where T is the maximum tangential stress at a point acting on the area passing through the second principal direction of the stress tensor

$$\tau_{xz} / \tau_{yz} = -\operatorname{tg} \varphi(x, y) \tag{1.5}$$

and φ is the angle made by the second principal direction with the x axis.

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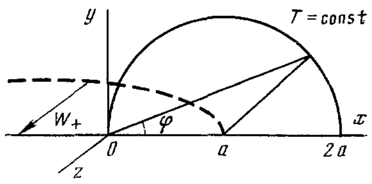

Fig. 1

In the elastic strain case it follows from the equilibrium equation

$$\partial\tau_{xz} / \partial x + \partial\tau_{yz} / \partial y = 0 \tag{1.6}$$

and (1.1), as well as from Hooke's law

$$\tau_{xz} = \mu\varepsilon_{xz}, \quad \tau_{yz} = \mu\varepsilon_{yz} \tag{1.7}$$

(μ is the shear modulus) that W, τ_{xz}, τ_{yz} are harmonic functions in the xy plane with the slit along the negative y half-axis.

The principal directions of the stress and strain tensors in an isotropic solid coincide for elastic strain.

2. Let us consider the solution of the elastic problem

$$W(x, y) = A\sqrt{r}\sin\theta/2, \quad r = \sqrt{(x-a)^2 + y^2}, \quad \theta = \operatorname{arc\,tg} y/(x-a) \tag{2.1}$$

where A, a are constants.

The harmonic function W (x, y) describes elastic displacements of points of the medium along the z axis; the slit apex is transferred to the point (a, 0) in the xy plane (Fig. 1). The solution (2.1) corresponds to the following boundary value problems: the boundary value of the harmonic function is given for y = 0, the parabolic shape of the slit is given for x < a

$$\begin{aligned} W(x, 0) &= A\sqrt{r}, \quad y = +0 \\ W(x, 0) &= -A\sqrt{r}, \quad y = 0 \end{aligned} \tag{2.2}$$

and the displacement W (x, 0) =0 for x > a.

The stresses and strains corresponding to (2.1)

$$\tau_{xz} = \mu\varepsilon_{xz} = -\frac{A\mu}{2\sqrt{r}}\sin\frac{\theta}{2}, \quad \tau_{yz} = \mu\varepsilon_{yz} = \frac{A\mu}{2\sqrt{r}}\cos\frac{\theta}{2} \tag{2.3}$$

exhibit an integrable singularity at the point (a, 0) and on the following boundary values on the y = 0 axis, τ_{xz} =0 for x > a and $\tau_{xz} \neq 0$ and assumes values on the slit edges which are equal in magnitude but opposite in sign; τ_{yz} =0 on the slit edges but $\tau_{yz} \neq 0$ for x > a.

Let us note that according to [1, 2] the magnitude of the maximum tangential stress T is constant on circles r =const. In particular, for r = a

$$T = A\mu / 2\sqrt{a} \tag{2.4}$$

Let us evaluate the discontinuity in the displacement for x = 0, y = 0 and the quantity (2.3) there

$$W_+ - W_- = 2A\sqrt{2} \tag{2.5}$$

The plus and minus signs refer to the upper and lower edges of the slit, while the quantity (2.5) determines the discontinuity in the displacement on the slit for r = a

$$\varepsilon_{xz} = \frac{\partial W}{\partial x} = \begin{cases} A/2\sqrt{a}, & y = +0, \quad x \leqslant 0 \\ A/2\sqrt{a}, & y = -0, \quad x \leqslant 0 \end{cases} \tag{2.6}$$

The property of the elastic solution remarked in (2.4) permits examination of the elastic–plastic problem. If the quantity T = T_e =const is taken as the elastic limit

$$T_e = \mu\Gamma_e \tag{2.7}$$

where Γ_e is the ultimate elastic value of the principal shear, and the radius of the plastic zone is taken equal to a, then the elastic distribution of the stresses and strains within a circle of radius a can be replaced by an elastic–plastic field of stresses and strains.

Let us find A from (2.4) and (2.7)

$$A = 2\Gamma_e\sqrt{a} = 2T_e\mu^{-1}\sqrt{a} \tag{2.8}$$

and let us introduce in conformity with (1.5)

$$\tau_{xz} = -T_e\sin\varphi, \quad \tau_{yz} = T_e\cos\varphi \tag{2.9}$$

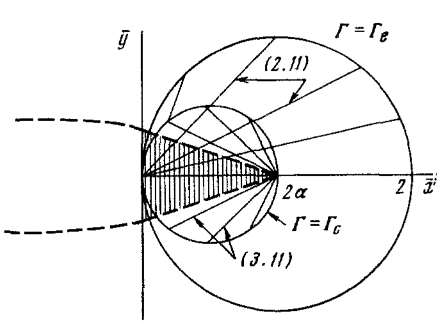

Fig. 2

The quantities (2.9) satisfy the plasticity conditions everywhere within the domain $r \leq a$. The equation for φ (x, y) follows from the equilibrium equation (1.6):

$$\cos\varphi \;\; \partial\varphi / dx + \sin\varphi\partial\varphi / \partial y = 0 \tag{2.10}$$

which has the characteristic

$$dy / dx = \mathrm{tg}\,\varphi \tag{2.11}$$

From the continuity of the stresses on the elastic plastic boundary there follows

$$\mathrm{tg}\,\varphi\,|_{r=a} = \mathrm{tg}\,\theta_0 / 2, \quad \theta_0 = \mathrm{arc\,tg} / (x - a)\,|_{r=a} \tag{2.12}$$

or

$$\varphi\,|_{r=a} = {}^1/_2\theta\,|_{r=a} \tag{2.13}$$

It follows from (2.11) and (2.13) that the stress is described by a centered field, by a fan of rectilinear characteristics passing through the origin and having the slope $\theta_0/2$. Hence the stresses (2.9)

$$\tau_{xz} = -T_e y\,(x^2 + y^2)^{-1/2}, \quad \tau_{yz} = T_e x\,(x^2 + y^2)^{-1/2} \tag{2.14}$$

satisfy the equilibrium equation, the plasticity condition, and the boundary conditions on the circle $r = a$.

The component is $\tau_{xz} = 0$ on the $y = 0$ axis in the plastic domain (Fig. 2) while $\tau_{yz} = T_e$ down to the singularity ($x = 0$, $y = 0$).

At this point, as φ varies in the range from $-\pi/2$ to $\pi/2$ (two subranges $(-\pi/2, 0)$ and $(\pi/2, 0)$ are two continuously interlocking fans of characteristics), the stress τ_{xz} varies between T_e and $-T_e$, while the stress τ_{yz} vanishes at the ends of the interval $(-\pi/2, \pi/2)$. The stresses receive the increments

$$\begin{array}{l} \Delta t_{xz} = -T_e\Delta\varphi\cos\varphi \\ \Delta\tau_{yz} = -T_e\Delta\sin\varphi \end{array}, \quad \Delta\tau_{xz} / \Delta\tau_{yz} = \mathrm{ctg}\,\varphi \tag{2.15}$$

upon passing from one characteristic to another so that the second principal directions of the stress tensors and the stress increments are orthogonal.

The principal direction of the stress tensor which held upon the onset of the plastic state is conserved at points of the plastic zone; the loading is "simple" [3]. Under these conditions, isotropic models of the strain (and the flow theory) type in the problem under consideration result in the corollary that not only the displacements in the elastic–plastic domain are continuous but so are the strain tensor components. Consequently, we obtain a solution with elevated smoothness in the neighborhood of the elastic–plastic boundary which does not follow from the formulation of the problem.

3. Let us take the following model of an elastic–plastic solid to determine the strain in the plastic zone:

1) As the plastic shear strains develop $\Gamma_e \leq \Gamma \leq \Gamma_c$, let us consider the stress and strain tensors coaxial (since the stress of the slipplane is conserved in an element).

2) Rupture can occur in some domain upon reaching $\Gamma - \Gamma_c$ so that the coaxiality condition can be rejected to be replaced by the condition in the rupture domain [4]. In conformity with the model, let us consider the domain $r \leq a$ in which $\Gamma \geq \Gamma_e$ can be realized. From the coaxiality condition

$$\varepsilon_{xz} / \varepsilon_{yz} = \tau_{xz} / \tau_{yz}' \tag{3.1}$$

and (1.2) and (2.14), there follows an equation [5] for:

$$x\partial\Gamma / \partial x + y\partial\Gamma / \partial y + \Gamma = 0 \tag{3.2}$$

It can be established that this equation is valid for the strain components ε_{xz}, ε_{yz} in the plastic domain. Integrating (3.2) and subjecting the general solution to the condition $\Gamma = \Gamma_e$ at $r = a$ (or on a line)

$$(x^2 - 2ax + y^2 = 0). \tag{3.3}$$

we find

$$\Gamma = 2\Gamma_e\bar{x} / (\bar{x}^2 + \bar{y}^2), \quad \bar{x} = x / a, \quad \bar{y} = y / a \tag{3.4}$$

The strain components ε_{xz}, ε_{yz} in the plastic domain are

$$\varepsilon_{xz} = \frac{\Gamma y}{\sqrt{\bar{x}^2 + \bar{y}^2}} = -\frac{2\Gamma_e \bar{x} y}{(\bar{x}^2 + \bar{y}^2)^{3/2}}, \quad \varepsilon_{yz} = \frac{2\Gamma_e \bar{x}^2}{(\bar{x}^2 + \bar{y}^2)^{3/2}} \tag{3.5}$$

Direct substitution can verify that the strain components satisfy the compatibility condition and the boundary conditions on the elastic–plastic boundary.

Evaluating the displacements in the plastic zone, we find

$$dW = \varepsilon_{xz} dx \neq \varepsilon_{yz} dy = \varepsilon_{xz} dx\,(1 + (\varepsilon_{xz} / \varepsilon_{xz})(dy / dx))$$

or $W = \text{const} = W|_{r=a}$ along the characteristic (2.11), along which

$$1 + (\varepsilon_{yz} / \varepsilon_{xz})(dy / dx) = 0$$

It follows from the expression for Γ in the plastic domain that the quantity is Γ =const on the circles:

$$(\bar{x} - \Gamma_e / \Gamma)^2 + \bar{y}^2 = (\Gamma_e / \Gamma)^2 \tag{3.6}$$

Along these circle

$$\Gamma = 2\Gamma_e / \bar{x}, \quad 0 < \bar{x} < 2$$

It hence follows that on going from $\Gamma = \Gamma_e$ to $\Gamma > \Gamma_e$ the point of maximum values of Γ is on the bisectrix $y = x$. If the change in Γ along the line y =const in the plastic zone is investigated, then it can be established that Γ varies between $\Gamma = \Gamma_e$ and the maximum achievable value of Γ on this line for $y = x$ and then decreases to $\Gamma = \Gamma_e$.

This means that if the problem of a moving slit is examined, then it is necessary to go over to equations for the unloading domain on the line $y = x$ [6].

If the domain in which $\Gamma = \Gamma_c$ is not introduced, then the solution (3.4) describes the singularity at the point $x=0$, $y=0$ near which the principal shear is (at $y = 0$, say)

$$\Gamma / \Gamma_e = 2 / x$$

Hence, let us introduce the domain $\Gamma = \Gamma_c$ and let us consider elements of the medium therein can be ruptured. It follows from (3.4) and (3.6) that the condition $\Gamma = \Gamma_c$ is satisfied on the circle

$$(\bar{x} - \alpha)^2 + \bar{y}^2 = \alpha^2, \quad \alpha = \Gamma_e / \Gamma_c < 1 \tag{3.7}$$

The stress distribution everywhere in the domain $r \le a$, including within the circle (3.7) as well, remains as before because of the equilibrium equation and the plasticity condition [see (2.14)]. To find the strain distribution, the "rupture" condition $\Gamma = \Gamma_c$ or

$$\sqrt{\varepsilon_{xz}^2 + \varepsilon_{yz}^2} = \Gamma_e \tag{3.8}$$

will be used together with the compatibility condition (1.1).

The condition (3.8) closes the system of equations in the rupture domain in place of the coaxiality condition (3.1) in the plastic domain.

Let us introduce ($y > 0$)

$$\varepsilon_{xz} = \partial W / \partial x = \Gamma_e \cos \psi_1, \quad \varepsilon_{yz} = \partial W / \partial y \tag{3.9}$$

and we obtain from the condition (1.1)

$$\cos \psi_1 \, \partial \psi_1 / \partial x + \sin \psi_1 \, \partial \psi_1 / \partial y = 0 \tag{3.10}$$

which agrees with (2.10) and has the characteristic

$$\partial y / \partial x = \operatorname{tg} \psi_1 \tag{3.11}$$

Here $\psi_1 = \psi + \pi/2$, i.e., characteristics (3.11) and (2.11) are orthogonal at points of the contour $\Gamma = \Gamma_c$, where $\varphi = \psi$ from the side of the elastic–plastic domain. The condition of continuity of the displacements would be satisfied on the contour $\Gamma = \Gamma_c$. The displacements are calculated from the elastic–plastic domain direction on this contour. It follows from (3.9) that W=const along the characteristic (3.11), i.e., it is a line of discontinuity for ψ_1=const, and $\psi_1 = \pi/2$ ($\psi = 0$). These facts permit the construction of a solution for the strain component in the form of two centered fields with common apex (singularity) at the point $(2\alpha, 0)$ (Fig. 2).

For y > 0 ($\pi \geq \psi_1 > \pi/2$) we have

$$\varepsilon_{xz} = \frac{\Gamma_c (\bar{x} - 2\alpha)}{\sqrt{(\bar{x} - 2\alpha)^2 + \bar{y}^2}}, \quad \varepsilon_{yz} = \frac{\Gamma_c y}{\sqrt{(\bar{x} - 2\alpha)^2 + \bar{y}^2}} \tag{3.12}$$

and for y > 0 ($0 \leq \psi_1 < \pi/2$)

$$\varepsilon_{xz} = \frac{\Gamma_c (\bar{x} - 2\alpha)}{\sqrt{(\bar{x} - 2\alpha)^2 + \bar{y}^2}}, \quad \varepsilon_{yz} = \frac{\Gamma_c \bar{y}}{\sqrt{(\bar{x} - 2\alpha)^2 + \bar{y}^2}} \tag{3.13}$$

and the displacements on both sides of the line y=0 on the segment $0 \leq x \leq 2\alpha$ are

$$W = \pm \Gamma_c a \sqrt{(\bar{x} - 2\alpha)^2 + y^2} \tag{3.14}$$

The plus and minus signs refer to points of the domain with ordinates $y \gtrless 0$, respectively.

The following graphical method of calculating the stresses and strains in the domain Γ_c =const can be mentioned. The value of the stress can be taken from the contour $r = a$ to the point (x, y) along the characteristic (2.11), and the value of the displacement and strain from the contour (3.7) along the characteristic (3.11).

Indicated in Fig. 2 is the shape which the slit has in the presence of an elastic–plastic hinge and with a description of the possible rupture upon reaching $\Gamma = \Gamma_c$. Thus, in the plastic domain $\Gamma_c \leq \Gamma \leq \Gamma_c$ the displacement is W=0 down to the point (2α, 0, $\alpha < 1$). The stress intensity is T_e on the discontinuous line. Opening of the slit to an angle equal to Γ_c at the vertex occurs on the section $0 < \bar{x} < 2\alpha$ so that the quantity $W_c = W_+ - W_-$ equals $4a\Gamma_e$ = at the origin. The value of this quantity can be used as a rupture criterion [2, 7]. The constancy of $T = T_e$ on this section can be interpreted as the constancy of the adhesion force intensity which retains the edges together with the opening of the slit at $W = W_c$.

The elastic–plastic solution constructed is of interest in that it permits analysis of the flow law when the coaxiality condition is spoiled: agreement between the principal directions of the stress and strain tensors holds up to the critical values $\Gamma = \Gamma_c$ in this problem, while the maximum tangential stress and the principal shear have the greatest values on different areas at $\Gamma = \Gamma_c$. This means that the flow law should be represented as a line in the space T, Γ, θ, where θ is the angle of the difference between the principal directions [3].

By studying the stress–strain state of elements on the line $|y|$ = const it is possible to indicate their sequential state along the line $|y| < a$ and back for the passage from elements ahead of the slit apex x > 0 to elements near the slit edges.

The simplest version $\Gamma_e = \Gamma_c$, an ideal "brittle" material, can be examined in the model of an elastic–plastic solid with the strength criterion $\Gamma = \Gamma_c$. In this case, the condition $\Gamma = \Gamma_c$ is reached on the circle $r = a$ and all the results for the "rupture" domain are extended to the circle $r = a$. This solution is suitable for a slowly moving slit where in each element near the slit ($y \leq a$) bypassed by the slit apex the following cycle holds: loading by the elastic law to $\Gamma_e = \Gamma_c$ and unloading by the elastic law. This solution is meaningful for an estimate of the size of the focus and energy in mountain socks and earthquakes.

To investigate the stability of the material in the domain $\Gamma = \Gamma_c$, let us evaluate the work of the stress increments on the strain increments [6]

$$\Delta\tau_{xz}\, \Delta\varepsilon_{xz} + \Delta\tau_{yz}\, \Delta\varepsilon_{yz} = T_e \Gamma_c \cos(\varphi - \psi) \Delta\varphi \Delta\psi$$

We will have $\varphi = \psi$ (stable equilibrium) on the boundary with the domain $\Gamma = \Gamma_c$ and $\psi = 0$ (unstable equilibrium) at the ends of the segment $0 < \bar{x} < 2\alpha$, where $\varphi = 0$, $\psi = \pi/2$ or $\varphi = \pi/2$. On this basis, the deduction can be made that compliance with the criterion $\Gamma = \Gamma_c$ in the unstable equilibrium state is necessary for the origination of rupture.

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